

# On Kossakowski construction of positive maps in matrix algebras

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## Abstract

We provide a further analysis of the class of positive maps proposed ten years ago by Kossakowski. In particular we propose a new parametrization which reveals an elegant geometric structure and an interesting interplay between group theory and a certain class of positive maps.

**Dedicated to Andrzej Kossakowski on his 75th birthday**

## 1 Introduction – a diagonal type positive maps

Ten year ago in a remarkable paper [1] Kossakowski provided a construction of a family of positive maps in matrix algebras  $M_n(\mathbb{C})$ . This construction reproduces many examples of positive maps already known in the literature. The maps from [1] belong to the following class: let  $\{e_0, \dots, e_{n-1}\}$  denotes an orthonormal basis in  $\mathbb{C}^n$  and let  $E_{ij} := |e_i\rangle\langle e_j|$ . Consider the linear map  $\Lambda : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  defined as follows

$$\Lambda(E_{ii}) = \sum_{j=0}^{n-1} a_{ij} E_{jj} , \quad \Lambda(E_{ij}) = -E_{ij} , \quad i \neq j . \quad (1)$$

where  $a_{ij}$  provides a set of complex parameters. In what follows we call the above maps *diagonal type maps*, since only diagonal elements  $E_{ii}$  are transformed in a non-trivial way. A map  $\Lambda$  is Hermitian, i.e.  $[\Lambda(X)]^\dagger = \Lambda(X^\dagger)$  iff  $a_{ij} \in \mathbb{R}$ . The basic question one poses is:

*what are conditions for  $a_{ij}$  which guarantee that  $\Lambda$  is a positive map.*

It is clear that a necessary condition is that all matrix elements  $a_{ij} \geq 0$ . Observe, that  $n \times n$  matrix  $A = [a_{ij}]$  with matrix elements  $[a_{ij}] \geq 0$  may be considered as a “classical” positive linear map  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Therefore, formula (1) provides a construction of a “quantum” positive map  $\Lambda$  out of the “classical” map  $A$  if “classical” conditions  $a_{ij} \geq 0$  are completed by a set of suitable “quantum” conditions. This problem is easily solvable for  $n = 2$ . One proves the following

**Proposition 1.** *If  $n = 2$ , then  $\Lambda$  is positive if and only if  $a_{ij} \geq 0$  and*

$$\sqrt{a_{00}a_{11}} + \sqrt{a_{01}a_{10}} \geq 1 . \quad (2)$$

*Moreover,  $\Lambda$  is completely positive if and only if  $a_{ij} \geq 0$  and  $a_{00}a_{11} \geq 1$ .*

The prescription (1) for  $\Lambda$  is so simple that it seems that for  $n > 2$  the corresponding additional conditions for  $a_{ij}$  are easy to find. Surprisingly, it is not the case and starting with  $n = 3$  the general problem is open. We stress that there is an essential difference between  $n = 2$  and  $n > 2$ . For  $n = 2$  all

positive maps are decomposable. It is no longer true for  $n > 2$ . And there are well known examples of indecomposable maps belonging to a general family (1).

Let us recall that a map  $\Lambda$  is positive iff for all rank-1 projectors  $P$  and  $Q$

$$\text{tr}[P\Lambda(Q)] \geq 0 . \quad (3)$$

Taking  $P = |x\rangle\langle x|$  and  $Q = |y\rangle\langle y|$  one has  $\langle x|\Lambda(|y\rangle\langle y|)|x\rangle \geq 0$  for all  $x, y \in \mathbb{C}^n$ . Using this definition one may prove the following

**Theorem 1** ([5]). *A map  $\Lambda$  defined in (1) is positive if and only if  $a_{ij} \geq 0$  and for all vectors  $x \in \mathbb{C}^n$*

$$\sum_{i=0}^{n-1} \frac{|x_i|^2}{B_i(x)} \leq 1 , \quad (4)$$

where

$$B_i(x) = |x_i|^2 + \sum_{j=0}^{n-1} a_{ij} |x_j|^2 . \quad (5)$$

Moreover,  $\Lambda$  is completely positive if and only if the matrix  $D = [d_{ij}]$  such that  $d_{ij} = -1$  for  $i \neq j$  and  $d_{ii} = a_{ii}$  is positive semi-definite.

We stress that an inequality (4) does not provide a solution to our problem. It is just a reformulation of the original definition of positivity for the special class of maps! One may easily check that for  $n = 2$  an inequality (4) reproduces condition (2). However, for  $n > 2$  we do not know how to translate the above inequality into the closed set of conditions upon the matrix elements  $a_{ij}$ .

## 2 Circulant matrices

Consider now a special case when  $a_{ij}$  defines a circulant matrix, i.e.  $a_{ij} = \alpha_{i-j \pmod n}$ . Actually, many well known examples of positive maps belongs to such class (e.g. reduction map, Choi map and its generalizations). We assume that  $\alpha_k \geq 0$  for  $k = 0, \dots, n-1$  and we denote the corresponding map by  $\Lambda[\alpha_0, \dots, \alpha_{n-1}]$ .

**Example 1.** For  $n = 2$  denoting  $a_{00} = a_{11} = \alpha_0 =: a$  and  $a_{01} = a_{10} = \alpha_1 =: b$  formula (2) reduces to

$$a + b \geq 1 . \quad (6)$$

Recall, that  $a = 0$  and  $b = 1$  corresponds to the reduction map  $R_2(X) = \mathbb{I}_2 \text{tr} X - X$ .

For a circulant matrix Theorem 1 reduces to the following

**Proposition 2.** *A map  $\Lambda[\alpha_0, \dots, \alpha_{n-1}]$  defined in (1) is positive if and only if for all vectors  $x \in \mathbb{C}^n$*

$$\sum_{i=0}^{n-1} \frac{|x_i|^2}{(\alpha_0 + 1)|x_i|^2 + \sum_{k=1}^{n-1} \alpha_k |x_{i+k}|^2} \leq 1 . \quad (7)$$

Moreover,  $\Lambda$  is completely positive if and only if  $\alpha_0 \geq n-1$ .

An inequality (7) is known as *circulant inequality* [6]. In particular taking  $|x_0| = \dots = |x_{n-1}|$  one finds the following necessary condition for positivity of  $\Lambda$

$$\alpha_0 + \alpha_1 + \dots + \alpha_{n-1} \geq n-1 . \quad (8)$$

Note, that the above condition is necessary but not sufficient. Actually, it is sufficient only for  $n = 2$  (see Example 1). For  $n = 3$  a full class of parameters  $\alpha_0 = a$ ,  $\alpha_1 = b$  and  $\alpha_2 = c$  satisfying circulant inequality (7) was derived in [4].

**Theorem 2.** ([4]) For  $n = 3$  a map  $\Lambda[a, b, c]$  is positive if and only if

1.  $a + b + c \geq 2$ ,
2. if  $a \leq 1$ , then  $bc \geq (1 - a)^2$ .

Moreover, being a positive map it is indecomposable if and only if

$$4bc < (2 - a)^2 . \quad (9)$$

$\Lambda$  is completely positive if and only if  $a \geq 2$ .

Hence, for  $n = 3$  a necessary condition  $a + b + c \geq 2$  is supplemented by an extra condition 3.

**Corollary 1.** If  $a > 1$ , then condition (8) is necessary and sufficient for positivity of  $\Lambda[a, b, c]$ .

**Remark 1.** For  $n > 3$  a full set of necessary and sufficient conditions for positivity of  $\Lambda[\alpha_0, \dots, \alpha_{n-1}]$  is not known.

### 3 Kossakowski construction

Let us define a set of Hermitian diagonal traceless matrices

$$F_\ell = \frac{1}{\sqrt{\ell(\ell+1)}} \left( \sum_{k=0}^{\ell-1} E_{kk} - \ell E_{\ell\ell} \right), \quad \ell = 1, \dots, n-1 . \quad (10)$$

These matrices span the Cartan subalgebra of  $su(n-1)$ . Moreover,  $\text{tr}(F_\alpha F_\beta) = \delta_{\alpha\beta}$ . Define a real  $n \times n$  matrix

$$a_{ij} := \frac{n-1}{n} + \sum_{\alpha, \beta=1}^{n-1} \langle e_i | F_\alpha | e_i \rangle R_{\alpha\beta} \langle e_j | F_\beta | e_j \rangle , \quad (11)$$

where  $R_{\alpha\beta}$  is an  $(n-1) \times (n-1)$  orthogonal matrix. Consider now a linear map  $\Lambda$  defined by (1) with  $a_{ij}$  defined by (11).

**Theorem 3** ([1]). For any orthogonal matrix  $R_{\alpha\beta}$  a linear map  $\Lambda$  is positive.

**Remark 2.** Actually Kossakowski provided more general construction [1]. However, in this paper we restrict our analysis to the special class of diagonal type maps corresponding to (11).

Due to the fact that  $F_\alpha$  is traceless for  $\alpha = 1, \dots, n-1$ , one finds

$$\sum_{i=1}^{n-1} a_{ij} = \sum_{j=1}^{n-1} a_{ij} = n-1 . \quad (12)$$

Moreover, since matrix elements  $a_{ij} \geq 0$  (it follows from Theorem 3) one finds that

$$\tilde{a}_{ij} := \frac{1}{n-1} a_{ij} , \quad (13)$$

defines a doubly stochastic matrix.

**Remark 3.** A map  $\tilde{\Lambda} := \frac{1}{n-1} \Lambda$  is unital trace preserving.

Consider now an inverse problem: suppose we are given a  $n \times n$  matrix  $[a_{ij}]$  such that  $[\tilde{a}_{ij}]$  is doubly stochastic. How to check whether  $a_{ij}$  is defined via (11)? The answer is given by the following

**Proposition 3** ([5]). *A matrix  $[a_{ij}]$  can be represented by (11) if and only if*

$$\sum_{k=0}^{n-1} a_{ik}a_{jk} = \delta_{ij} + n - 2 , \quad (14)$$

for  $i, j = 0, \dots, n-1$ .

Define

$$b_{ij} := a_{ij} - 1 , \quad (15)$$

that is,

$$b_{ij} = \sum_{\alpha, \beta=1}^{n-1} \langle e_i | F_\alpha | e_i \rangle R_{\alpha\beta} \langle e_j | F_\beta | e_j \rangle - \frac{1}{n} , \quad (16)$$

One easily proves

**Proposition 4.** *A matrix  $[a_{ij}]$  satisfies (17) if and only if matrix  $[b_{ij}]$  satisfies*

$$\sum_{k=0}^{n-1} b_{ik}b_{jk} = \delta_{ij} , \quad (17)$$

for  $i, j = 0, \dots, n-1$ , i.e.  $[b_{ij}]$  is an orthogonal matrix.

Note, that if  $[b_{ij}]$  defines an orthogonal matrix, then  $|b_{ij}| \leq 1$  and hence  $a_{ij} = b_{ij} + 1 \geq 0$ .

**Corollary 2.** *A map  $\Lambda$  defined in (1) is positive if the corresponding  $b_{ij}$  defines  $n \times n$  orthogonal matrix such that*

$$\sum_{i=0}^{n-1} b_{ij} = \sum_{j=0}^{n-1} b_{ij} = -1 . \quad (18)$$

It is clear that formula (16) provides an embedding of  $O(n-1)$  into  $O(n)$ , i.e. an orthogonal matrix  $R_{\alpha\beta}$  from  $O(n-1)$  is mapped into an orthogonal matrix  $b_{ij}$  from  $O(n)$ .

Now, we provide a geometric interpretation of Kossakowski construction. Let  $\{\mathbf{b}^{(0)}, \dots, \mathbf{b}^{(n-1)}\}$  be an orthonormal basis in  $\mathbb{R}^n$  such that

$$(\mathbf{b}^{(i)}, \mathbf{e}) = -\frac{1}{\sqrt{n}} , \quad (19)$$

where  $(\mathbf{a}, \mathbf{b})$  denotes the canonical inner product in  $\mathbb{R}^n$  and

$$\mathbf{e} = \frac{1}{\sqrt{n}}(1, \dots, 1) . \quad (20)$$

Let us define

$$b_{ij} := \mathbf{b}_j^{(i)} . \quad (21)$$

Clearly,  $[b_{ij}]$  defines an orthogonal matrix. Moreover, (19) guarantees (18).

**Corollary 3.** *Any Kossakowski map is uniquely defined by an arbitrary orthonormal basis  $\{\mathbf{b}^{(0)}, \dots, \mathbf{b}^{(n-1)}\}$  satisfying (19).*

**Corollary 4.** *If  $[b_{ij}]$  defines a Kossakowski map, then  $[b_{i\pi(j)}]$  defines another Kossakowski map for an arbitrary permutation  $\pi \in S_n$ .*

Let  $\Sigma_{\mathbf{e}}$  denote an  $(n-1)$ -dimensional hyperplane in  $\mathbb{R}^n$  orthogonal to vector  $\mathbf{e}$ . Let  $\{\mathbf{f}^{(1)}, \dots, \mathbf{f}^{(n-1)}\}$  be an arbitrary orthonormal basis in  $\Sigma_{\mathbf{e}}$ . An example of such a basis is provided by

$$\mathbf{f}_i^{(\alpha)} = \langle e_i | F_\alpha | e_i \rangle , \quad (22)$$

where  $F_\alpha$  are defined in (10). Clearly,  $\{\mathbf{f}^{(0)} := \mathbf{e}, \mathbf{f}^{(1)}, \dots, \mathbf{f}^{(n-1)}\}$  defines an orthonormal basis in  $\mathbb{R}^n$ . Consider now an orthogonal operator  $\mathbf{R}$  such that its matrix representation in the basis  $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \dots, \mathbf{f}^{(n-1)}\}$  has the following form

$$\mathbf{R}_{00} = -1 , \quad \mathbf{R}_{0k} = \mathbf{R}_{k0} = 0 , \quad \mathbf{R}_{ij} = R_{ij} . \quad (23)$$

It is clear that  $\mathbf{R}$  represents rotation (or pseudo-rotation) around  $\mathbf{e}$ .

**Proposition 5.** *Let  $[b_{ij}]$  be the matrix representation of  $\mathbf{R}$  in the canonical basis in  $\mathbb{R}^n$ . Then  $b_{ij}$  satisfy (18).*

Proof: denote by  $\{\mathbf{e}_0, \dots, \mathbf{e}_{n-1}\}$  the canonical basis and let

$$\mathbf{f}^{(i)} = \sum_{j=0}^{n-1} S_{ij} \mathbf{e}_j . \quad (24)$$

One has

$$b = S^T \mathbf{R} S . \quad (25)$$

and hence

$$\sum_{i=0}^{n-1} b_{ij} = \sum_{k,l=0}^{n-1} \sum_{i=0}^{n-1} S_{ki} \mathbf{R}_{kl} S_{lj} = \sum_{i=0}^{n-1} S_{0i} \mathbf{R}_{00} S_{0j} + \sum_{\alpha,\beta=1}^{n-1} \sum_{i=0}^{n-1} S_{\alpha i} R_{\alpha\beta} S_{j\beta} = -1 , \quad (26)$$

due to

$$S_{0i} = \frac{1}{\sqrt{n}} , \quad \sum_{i=0}^{n-1} S_{\alpha i} = 0 , \quad i = 0, 1, \dots, n-1 ; \quad \alpha = 1, \dots, n-1 . \quad (27)$$

In particular if  $\mathbf{f}^{(i)}$  are defined *via* (22), then (25) reproduces (16).

Consider now the following symmetric set of  $n$  vectors  $\{\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(n-1)}\}$  in  $\Sigma_{\mathbf{e}}$  defined by:

1. they have the same length,
2. the angle ' $\phi_n$ ' between arbitrary two vectors is the same.

One proves that

$$\cos \phi_n = -\frac{1}{n-1} . \quad (28)$$

**Remark 4.** *Actually, a set of  $n$  vectors  $\{\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(n-1)}\}$  in  $\mathbb{R}^{n-1}$  satisfying the above conditions is called an equiangular frame [7].*

**Proposition 6.** *Vectors*

$$\mathbf{b}^{(i)} := \mathbf{g}^{(i)} - \frac{1}{\sqrt{n}} \mathbf{e} ,$$

*such that  $|\mathbf{g}^{(i)}|^2 = 1 - \frac{1}{n}$ , define an orthonormal basis in  $\mathbb{R}^n$  and satisfy (19).*

Consider now a special case when the matrix  $[a_{ij}]$  defined in (11) is circulant. Formula (12) implies

$$\alpha_0 + \dots + \alpha_{n-1} = n-1 . \quad (29)$$

In this case Proposition 4 reduces to

**Proposition 7.** *A circulant matrix  $a_{ij} = \alpha_{i-j}$  satisfies (17) if and only if*

$$\sum_{k=0}^{n-1} \alpha_{i-k} \alpha_{j-k} = \delta_{ij} + n - 2 , \quad (30)$$

for  $i, j = 0, \dots, n-1$ .

Introducing

$$\beta_i = \alpha_i - 1 , \quad (31)$$

one finds

$$\beta_0 + \dots + \beta_{n-1} = -1 , \quad (32)$$

together with

$$\sum_{k=0}^{n-1} \beta_{i-k} \beta_{j-k} = \delta_{ij} , \quad (33)$$

for  $i, j = 0, \dots, n-1$ . Clearly,  $b_{ij} = \beta_{i-j}$  defines a circulant orthogonal matrix satisfying an additional constraint (32).

## 4 Examples

**Example 2.** *For  $n = 2$  one has  $F_1 = \frac{1}{\sqrt{2}} \sigma_z$  and  $R = \pm 1$ , and hence one easily finds*

$$R = 1 \rightarrow [a_{ij}] = \mathbb{I}_2 ; \quad R = -1 \rightarrow [a_{ij}] = \sigma_x . \quad (34)$$

**Example 3.** *For  $n = 3$*

$$F_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad F_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} , \quad (35)$$

and

$$[R_{\alpha\beta}] = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} . \quad (36)$$

Interestingly, in this case one finds that the matrix  $[a_{ij}]$  is circulant. Denoting  $a := a_{00}, b := a_{01}$  and  $c := a_{02}$  one obtains

$$\begin{aligned} a &= \frac{2}{3}(1 + \cos \phi) , \\ b &= \frac{1}{3}(2 - \cos \phi - \sqrt{3} \sin \phi) , \\ c &= \frac{1}{3}(2 - \cos \phi + \sqrt{3} \sin \phi) . \end{aligned} \quad (37)$$

Let us observe that introducing  $\tilde{a} = a - 1, \tilde{b} = b - 1$  and  $\tilde{c} = c - 1$  the above family of maps is uniquely characterized by a circulant orthogonal matrix

$$[b_{ij}] = \begin{pmatrix} \tilde{a} & \tilde{b} & \tilde{c} \\ \tilde{c} & \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{c} & \tilde{a} \end{pmatrix} , \quad (38)$$

with  $\tilde{a} + \tilde{b} + \tilde{c} = -1$ . Interestingly, the well known maps: Choi maps  $\Lambda[1, 1, 0]$ ,  $\Lambda[1, 0, 1]$  and the reduction map  $\Lambda[0, 1, 1]$  have the following representation in terms of the matrix  $[b_{ij}]$ :

$$\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} ; \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad (39)$$

that is, up to a sign they correspond to circulant permutation matrices.

**Remark 5.** For  $n = 2$  and  $n = 3$  all Kossakowski maps are characterized by a circulant matrix  $[a_{ij}]$ . It is no longer true for  $n > 3$ .

**Remark 6.** Let us observe that parameters  $a, b, c$  defined in (37) are compatible with Theorem 2. Note, that maps defined via (37) belong to the boundary of a set of positive maps defined by two equalities in conditions 1. and 2. of Theorem 2, that is,

$$a + b + c = 2 ; \quad bc = (1 - a)^2 . \quad (40)$$

Detailed analysis of the structure of these maps was performed in [2].

**Example 4.** For  $n = 4$  one has the following circulant orthogonal  $[b_{ij}]$  matrix:  $\tilde{a} = b_{00}$ ,  $\tilde{b} = b_{01}$ ,  $\tilde{c} = b_{02}$  and  $\tilde{d} = b_{03}$  satisfying

$$\tilde{a} + \tilde{b} + \tilde{c} + \tilde{d} = -1 . \quad (41)$$

Orthogonality conditions imply

$$\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 = 1 , \quad \tilde{a}\tilde{c} + \tilde{b}\tilde{d} = 0 , \quad (\tilde{a} + \tilde{c})(\tilde{b} + \tilde{d}) = 0 . \quad (42)$$

Therefore, we have two classes of admissible parameters  $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$  constrained by

$$\tilde{a} + \tilde{b} + \tilde{c} + \tilde{d} = -1 , \quad \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 = 1 , \quad \tilde{b} + \tilde{d} = 0 , \quad (43)$$

and

$$\tilde{a} + \tilde{b} + \tilde{c} + \tilde{d} = -1 , \quad \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 = 1 , \quad \tilde{a} + \tilde{c} = 0 , \quad (44)$$

Equivalently, the above conditions may be rewritten as follows

$$\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 = 1 , \quad \tilde{a} + \tilde{c} = -1 , \quad \tilde{b} + \tilde{d} = 0 , \quad (45)$$

and

$$\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 = 1 , \quad \tilde{a} + \tilde{c} = 0 , \quad \tilde{b} + \tilde{d} = -1 . \quad (46)$$

They describe two circles: the intersection of 3D sphere with two planes. Again, characteristic well known maps  $\Lambda[1, 1, 1, 0]$ ,  $\Lambda[1, 1, 0, 1]$ ,  $\Lambda[1, 0, 1, 1]$  and  $\Lambda[0, 1, 1, 1]$  (up to a sign) correspond to circulant permutation matrices:

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

**Example 5.** For  $n = 5$  one has the following circulant orthogonal  $[b_{ij}]$  matrix:  $\tilde{a} = b_{00}$ ,  $\tilde{b} = b_{01}$ ,  $\tilde{c} = b_{02}$ ,  $\tilde{d} = b_{03}$  and  $\tilde{e} = b_{04}$  satisfying

$$\tilde{a} + \tilde{b} + \tilde{c} + \tilde{d} + \tilde{e} = -1 . \quad (47)$$

Orthogonality conditions imply

$$\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 + \tilde{e}^2 = 1 , \quad \tilde{a}\tilde{e} + \tilde{b}\tilde{a} + \tilde{c}\tilde{b} + \tilde{d}\tilde{c} + \tilde{e}\tilde{d} = 0 . \quad (48)$$

One easily checks that the remaining orthogonality conditions are not independent from (47) and (48). The corresponding set of admissible parameters  $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}\}$  is 2-dimensional but its shape is not very transparent (47) and (48).

## 5 Circulant case — a complementary parametrization

Let us recall that if  $a_{ij} = \alpha_{i-j}$  defines a circulant matrix, then its eigenvalues are given by

$$\lambda_k = \sum_{l=0}^{n-1} \omega^{-kl} \alpha_l, \quad k = 0, \dots, n-1, \quad (49)$$

and the corresponding eigenvectors read

$$\mathbf{x}_k = (1, \omega^k, \omega^{2k}, \dots, \omega^{(n-1)k})^T, \quad (50)$$

where  $\omega = e^{2\pi i/n}$ . Two sets  $\{\alpha_0, \dots, \alpha_{n-1}\}$  and  $\{\lambda_0, \dots, \lambda_{n-1}\}$  are related by the discrete Fourier transform. Note that

$$\lambda_0 = \alpha_0 + \dots + \alpha_{n-1} = n-1. \quad (51)$$

Consider now a circulant orthogonal matrix  $b_{ij} = \beta_{i-j}$  with  $\beta_k = \alpha_k - 1$ . The corresponding eigenvalues  $\mu_k$  of  $[b_{ij}]$  are defined by

$$\mu_0 = \lambda_0 - n = -1, \quad \mu_\alpha = \lambda_\alpha, \quad \alpha = 1, \dots, n-1. \quad (52)$$

Now, since  $[b_{ij}]$  is orthogonal one has  $|\mu_k| = 1$  and hence

**Proposition 8.** *Real parameters  $\{\alpha_0, \dots, \alpha_{n-1}\}$  satisfy (30) if and only if  $|\lambda_\alpha| = 1$  for  $\alpha = 1, \dots, n-1$ .*

This way we obtain a new parametrization of a set of admissible circulant matrices  $[a_{ij}]$  by phases of  $\lambda_\alpha = e^{i\phi_\alpha}$ . Due to  $\lambda_k = \lambda_{n-k}^*$  one has two cases:

1. if  $n = 2m + 1$ , then we have  $m$  independent phases  $\lambda_1 = e^{i\phi_1}, \dots, \lambda_m = e^{i\phi_m}$ .
2. if  $n = 2m + 2$ , then we have  $m$  independent phases  $\lambda_1 = e^{i\phi_1}, \dots, \lambda_m = e^{i\phi_m}$  and one real parameter  $\lambda_{m+1} = \pm 1$ .

**Example 6.** *For  $n = 3$  putting  $\lambda_1 = e^{i\phi} = \lambda_2^*$  one finds*

$$\begin{aligned} a &= \frac{1}{3}(2 + \lambda_1 + \lambda_1^*) = \frac{2}{3}(1 + \cos \phi), \\ b &= \frac{1}{3}(2 + \omega \lambda_1 + \omega^* \lambda_1^*) = \frac{1}{3}(2 - \cos \phi - \sqrt{3} \sin \phi), \\ c &= \frac{1}{3}(2 + \omega^* \lambda_1 + \omega \lambda_1^*) = \frac{1}{3}(2 - \cos \phi + \sqrt{3} \sin \phi), \end{aligned} \quad (53)$$

due to  $\omega = e^{2\pi i/3} = \frac{1}{2}(-1 + i\sqrt{3})$ . This reproduces result of Example 3.

**Example 7.** *For  $n = 4$  if  $\lambda_1 = e^{i\phi} = \lambda_3^*$  and  $\lambda_2 = 1$  one finds*

$$a = \frac{1}{2}(2 + \cos \phi), \quad b = \frac{1}{2}(1 - \sin \phi), \quad c = \frac{1}{2}(2 - \cos \phi), \quad d = \frac{1}{2}(1 + \sin \phi), \quad (54)$$

and similarly if  $\lambda_1 = e^{i\psi} = \lambda_3^*$  and  $\lambda_2 = -1$  one has

$$a = \frac{1}{2}(1 + \cos \psi), \quad b = \frac{1}{2}(2 - \sin \psi), \quad c = \frac{1}{2}(1 - \cos \psi), \quad d = \frac{1}{2}(2 + \sin \psi). \quad (55)$$

Note, that for  $\lambda_2 = 1$  one has  $b + d = 1$ , whereas for  $\lambda_2 = -1$  one has  $b + d = 2$ . This way we reproduced two classes from Example 4.

**Corollary 5.** *It is therefore clear that*



1. if  $n = 2m + 1$ , then a set of admissible parameters defines  $m$ -dimensional torus  $\mathbb{T}_m$ . Note that  $O(n - 1) = O(2m)$  and a single torus  $\mathbb{T}_m$  corresponds to a maximal commutative subgroup of  $SO(2m)$ .
2. if  $n = 2m + 2$ , we have two  $m$ -dimensional tori  $\mathbb{T}_m$  and  $\mathbb{T}'_m$ . Torus  $\mathbb{T}_m$  corresponds to a maximal commutative subgroup of  $SO(2m + 1)$  whereas  $\mathbb{T}'_m$  is defined by composing  $\mathbb{T}_m$  with reflection, that is,  $g \in \mathbb{T}'_m$  iff  $-g \in \mathbb{T}_m$  (cf. [8]).

**Corollary 6.** Positive maps  $\Lambda[\alpha_0, \dots, \alpha_{n-1}]$  are invertible. It follows from the fact that

$$|\det[a_{ij}]| = |\lambda_0 \dots \lambda_{n-1}| = n - 1 \neq 0 . \quad (56)$$

Note, however, that the inverse  $\Lambda^{-1}[\alpha_0, \dots, \alpha_{n-1}]$  is no longer positive.

## 6 Conclusions

We analyzed a class of positive maps introduced by Kossakowski [1]. It turns out that these maps display interesting geometric features. In particular its maximal commutative subset —  $\Lambda[\alpha_0, \dots, \alpha_{n-1}]$  — corresponding to circulant matrices  $[a_{ij}]$  is parameterized by tori which defines maximal commutative subgroups of the orthogonal group. For further properties of these maps like (in)decomposability and/or optimality see also [8]. It is clear that *via* Choi-Jamiołkowski isomorphism can one provide a similar analysis in terms of entanglement witnesses (see [9] for the recent review).

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